Modeling Perfectly Uniform Fluted Columns from the Difference of a Base Cylinder and n Aligned Cylinders

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1 Introduction

Columns are well used in many different architectural styles and throughout history. From the Parthenon of Ancient Greece to the U.S Capitol, we have been using columns as both a source of beauty and structural integrity for thousands of years. Ancient civilizations carefully carved them out of large stone blocks to fit their desired aesthetics[1]. Today, we can construct them digitally using 3D modelling software. In this paper I present one way of creating the shaft of a classic Greek column similar to figure 1 by taking the difference of a given base cylinder of any radius, and any number $n \geq 4$ carefully sized and positioned cutting cylinders. I leave the creation of the capital and base to the more artistically aligned.

Figure 1: The historic Cincinnati Gas and Electric Co. building, in Greek Doric style, Cincinnati, Ohio.[2]

2 Theoretical Analysis

2.1 Desired Outcome

Ignoring entasis, we can reduce the problem to a two-dimensional one. This will simply our calculations. If we can form the two-dimensional base of the shaft, then extending into 3 dimensions is as simple as extending it outward in the Z direction.

First, we identify approximately what our desired base should look like. From figure 1, one description of the shaft is a cylinder, out of which part of a circular pattern of cylinders have been carved out. We can model a top down view with 2D graphing software as shown in figure 2. Coincidentally, the final base shape after the difference operation can also be seen in figure 2 as the inner region delineated by the red curves.

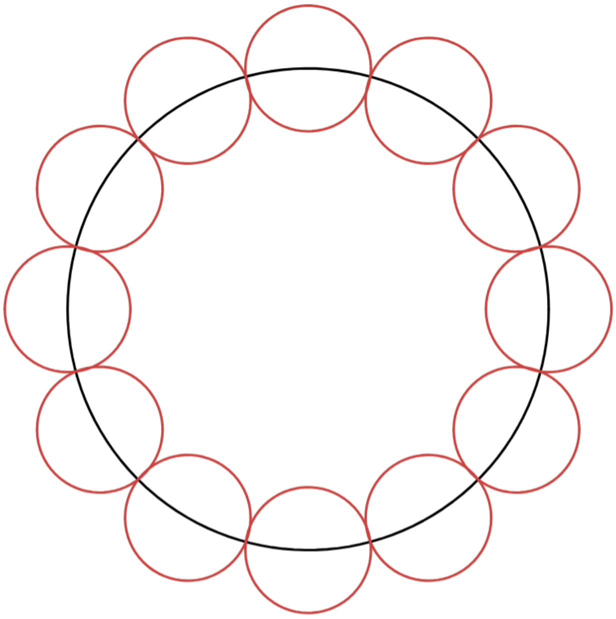

Figure 2: Original cylinder (in black), and the circular pattern of cylinders (in red) that will be cut from it.

2.2 Determining the Cutting Cylinder Radius

Given the radius of the base cylinder r, and given the number of cutting cylinders n that we want cut out (based on aesthetic preference), we need to compute a suitable radius, k, for the cutting cylinders. We need to pick k such that all of the base path is covered and, after the difference operation, will create uniform flutes in the column.

Back in two dimensions, if we consider only the first cutting circle whose center is along the positive x-axis, the following equation describes it:

$$(x - r)^2 + y^2 = k^2$$

To ensure every flute is uniform, we calculate the arc angle corresponding to the portion of base circle arc that each cutting circle will contain.

$$\theta = \frac{2\pi}{n}$$

From this, we can calculate the points P_1 and P_2 that define the beginning and end of the arc that is to be contained by the first cutting circle as:

$$P_1 = (r * cos(\frac{\theta}{2}), r * sin(\frac{\theta}{2})) = (r * cos(\frac{\pi}{n}), r * sin(\frac{\pi}{n}))$$

$$P_2 = (r * cos(-\frac{\theta}{2}), r * sin(-\frac{\theta}{2})) = (r * cos(-\frac{\pi}{n}), r * sin(-\frac{\pi}{n}))$$

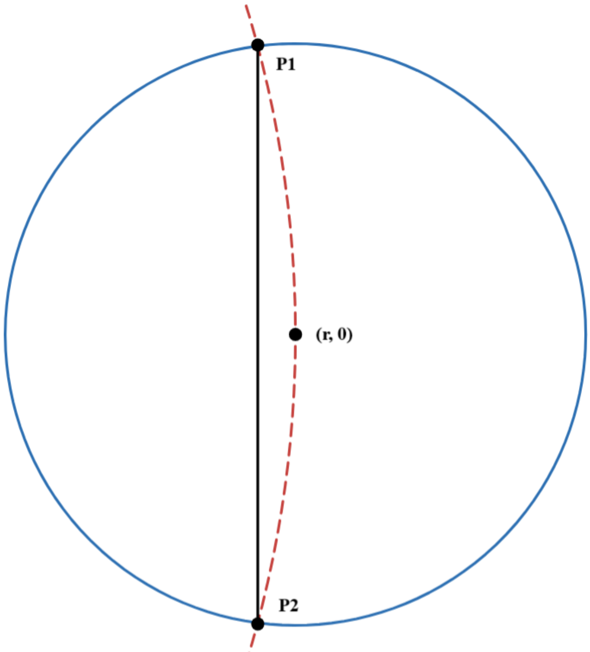

Figure 3: Possible chord formed by P_1 and P_2 (in black) in the first cutting circle. The base circle path is shown in red.

From here, we can calculate the radius, k, of the cutting circle by finding the length of the hypotenuse of the right triangle formed by the points $(r, 0)$, P_1, and $(r - P_{1x}, 0)$. From the Pythagorean theorem we have:

$$(r - P_{1x})^2 + (P_{1y})^2 = k^2$$

$$(r - r * cos(\frac{\pi}{n}))^2 + (r * sin(\frac{\pi}{n}))^2 = k^2$$

Solving for k:

$$r^2 - 2r^2 cos(\frac{\pi}{n}) + r^2 cos^2(\frac{\pi}{n}) + r^2 sin^2(\frac{\pi}{n}) = k^2$$

$$r^2(1 - 2cos(\frac{\pi}{n}) + cos^2(\frac{\pi}{n}) + sin^2(\frac{\pi}{n})) = k^2$$

From the Pythagorean identity:

$$r^2(1 - 2cos(\frac{\pi}{n}) + 1) = k^2$$

$$k = \pm\sqrt{r^2(2 - 2cos(\frac{\pi}{n}))}$$

$$k = \pm r * \sqrt{2 - 2cos(\frac{\pi}{n})}$$

Radius must be positive:

$$k = r * \sqrt{2 - 2cos(\frac{\pi}{n})}$$

Now that we have a suitable k, we can be certain that there exists a suitable position for each circle such that we can create uniform flutes after a difference operation. However, we see that for small values of n, k is greater than or equal to than r:

$$n = 1, k = r * \sqrt{2 - 2cos(\pi)}; k = r * \sqrt{4}; k = 2r$$

...

$$n = 3, k = r * \sqrt{2 - 2cos(\frac{\pi}{3})}; k = r * \sqrt{1}; k = r$$

It does not make sense for k to be larger than or equal to r. This would at minimum cause the center of the base to be cut out, an odd result. In practice, if positioned as described in section 2.3, for $n \leq 3$ the entire base would be removed after the difference operation, leaving no column at all. We therefore subject n to the following restriction:

$$n \geq 4$$

2.3 Positioning of Cutting Cylinders

In section 2.2 we found the ideal radius of a cutting cylinder, k, and saw the equation for the first cutting circle, c_0. Here we find the general equation for the circles $c_0, c_1, ..., c_{n-1}$. From section 2.2, we found that a cutting circle with radius k contains a portion of arc (with arc angle θ) from the base circle.

As part of our desired outcome we want each cutting circle to contain an equal yet unique section of base circle arc. In c_0, this section is defined by P_1 and P_2, an arc with arc angle θ, with the center of c_0 along it. To find the center of c_1, we simply rotate the center of c_0, $(r, 0)$, by θ.

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r * cos\theta \\ r * sin\theta \end{bmatrix}$$

We can find the center of any cutting circle c_i, by rotating the center of c_0 by the ith multiple of θ:

$$\begin{bmatrix} cos(i\theta) & -sin(i\theta) \\ sin(i\theta) & cos(i\theta) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r * cos(i\theta) \\ r * sin(i\theta) \end{bmatrix}$$

It follows that the equation for any cutting circle c_i is thus:

$$(x - r * cos(i\theta))^2 + (y - r * sin(i\theta))^2 = k^2$$

2.4 Optional Size Buffer

As described above and visible in figure 1, we size and position the cutting cylinders so the entire base circle path is filled and there is no overlap in the base circle arcs contained by these cylinders. This produces a sharp edge at the ends of each flute. To reduce floating point errors around the edges, or to improve aesthetic, I introduce a third parameter, v. This parameter is added to k to grow or shrink the resulting cutting cylinder radius, so that the new equation for any cutting circle c_i is:

$$(x - r * cos(i\theta))^2 + (y - r * sin(i\theta))^2 = (k + v)^2$$

Small positive values of v can help reduce artifacts where the cutting cylinders meet along the base circle path due to floating point errors. Larger values of v can be used to create shallower flutes, if desired. Negative values of v will leave a gap around the base path that creates blunted edges, if a sharp edge is not desired. A negative v needs to be small enough to counteract the overlap among cutting cylinders, or the blunted edge will be disconnected from the main shaft. This is especially true for small n where there is larger overlap among the cutting cylinders.

3 Algorithm

Computing the positions for each cutting cylinder based on the equation in 2.3 can be tedious, especially for large n. Here I present a simple algorithm to create a column based on r, n, v, and h, where h is the height of the column. The other variables are discussed above.

Algorithm 1 Create Column

Require: $r > 0, n \geq 4, h > 0$

```
function CREATE_COLUMN(r, n, v, h)
    base ← create_cylinder(r, h)
    theta ← 2π/n
    k ← r * sqrt(2 − 2 * cos(π/n))

    for i ← 0 to N − 1 do
        c_i ← create_cylinder(k + v, h + 1)        ▷ +1 to ensure full vertical cut
        set_position(c_i, (r * cos(i * theta), r * sin(i * theta), 0))
        difference_op(base, c_i)
        destroy_cylinder(c_i)
    end for
end function
```

4 Results

These example columns have been generated using Blender, a 3D modelling software, using an implementation of the algorithm described in section 3.

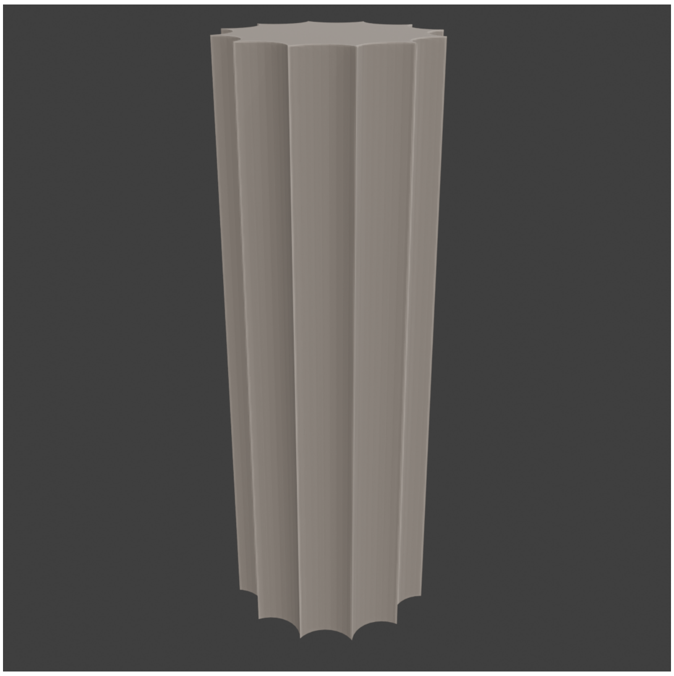

Figure 4: Column generated from $r = 4, n = 12, v = 0.05, h = 20$

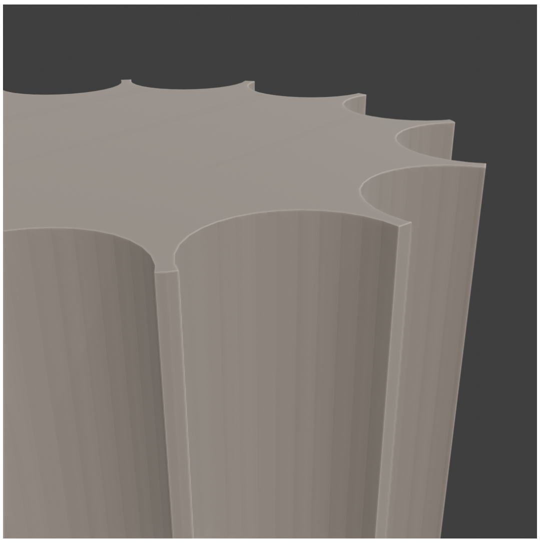

Figure 5: Column generated from $r = 4, n = 12, v = -0.08, h = 20$. This figure highlights the blunted edges that occur for negative values of v, as described in section 2.4

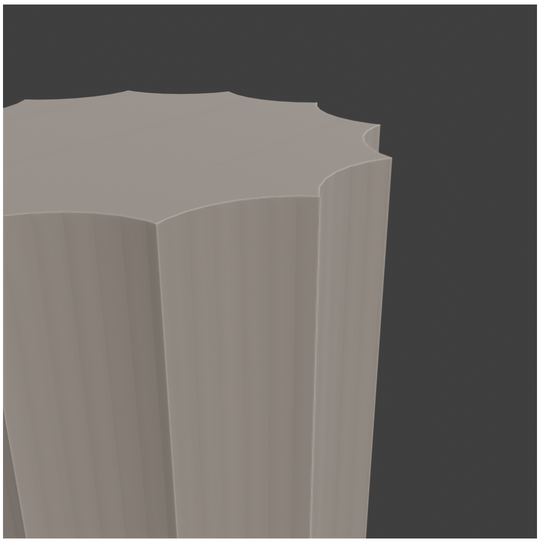

Figure 6: Column generated from $r = 4, n = 12, v = 0.3, h = 20$. This figure highlights the shallow flutes that occur for larger values of v.

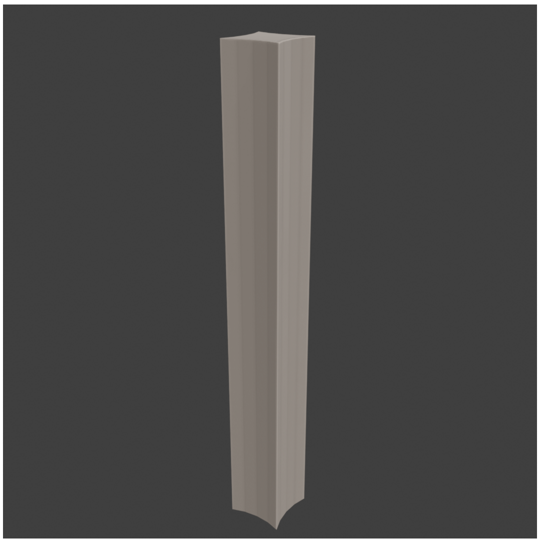

Figure 7: Column generated from $r = 4, n = 4, v = 0.01, h = 20$. Large parts of the base cylinder are cut when small n are used.

Figure 8: Column generated from $r = 4, n = 4, v = 0.05, h = 30$. Complete with Doric order capital and base added.

5 Discussion

One point of discussion is that the naive approach to finding k, using half of the circumference of the base circle divided by n, produces a very close approximation to the true value of k, often with error smaller than typical values of v. This is due to the chord P_1P_2 being a close approximation to arc P_1P_2 as seen in figure 3. This approximation is likely unnecessary in the vast majority of use cases, but if there is a need to create massive amounts of columns with varying parameters, the naive approach to finding k may be slightly faster.

We see from figure 7 that small values of n produce thinner columns. This is due to the large radius k required to cover the base circle path, which ends up cutting out the majority of the base cylinder. For $n < 4$, this effect results in the entire base cylinder being deleted. We can quantify this effect as the ratio of k to r. As n increases, $\frac{k}{r}$ decreases, and the radius of the remaining uncut concentric cylinder in the original base cylinder increases and approaches r.

6 Conclusion

Modelling fluted columns is a seemingly simple task, yet as we see in this paper, it turns out to be quite challenging to do precisely. Even when modelling by hand, choosing an incorrect k can easily result in broken uniformity by leaving too much or too little space left for the final flute. While such an imperfection may be insignificant in a larger scene, we need not settle any longer. Unlike the Greeks, in the digital world we have the capacity to make perfect columns. We should take advantage of that capacity.

References

[1] "Column," *Encyclopædia Britannica*. Available: `https://www.britannica.com/technology/column-architecture` [accessed Jul. 17, 2024].

[2] D. Jensen, "Doric order," *Encyclopædia Britannica*. Available: `https://www.britannica.com/technology/Doric-order#/media/1/169486/114687` [accessed Jul. 17, 2024].